Distribution of Eddy Viscosity and Mixing Length in Smooth Tubes

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Profiles of eddy viscosity and Prandtl mixing length in fluids flowing steadily and isothermally in smooth tubes have been calculated from the velocity data of several investigators for Reynolds numbers between 1.2 × 10³ and 3.2 × 10⁶. In the transition range unusually high values of eddy viscosity and mixing length are obtained in some portions of the stream. In the fully turbulent range the effect of Reynolds number is small and the mixing length tends toward zero at the center of the tube. The parameters for turbulent flow between parallel plates have been correlated through the concept of an equivalent tube. The results are of importance in designing equipment for heat and mass transfer and mixing.

When dealing with processing equipment, one often finds it necessary or highly desirable to be able to predict the rate of heat, mass, or momentum transfer at a particular point in a moving fluid. In general, this requires some knowledge of the relationship between the designated flux and the corresponding potential gradient at the spot in question. If the flow is truly viscous, a direct solution for the desired rate of transfer is sometimes possible. When the flow is turbulent to any extent, however, empirical information is ultimately necessary, even in ducts having simple cross-sectional shapes.

Progress in predicting mixing phenomena as well as local rates of heat and mass transfer has not been so rapid as might be expected. One of the principal reasons for this seems to be the lack of sufficient consistent information about eddy viscosities in elementary conduits such as smooth tubes of circular cross section. In spite of the relatively large number of velocity distributions available in the literature, there have been but few attempts to translate these into profiles of eddy viscosities or mixing lengths. In particular, the effect of Reynolds number on the radial distribution of these parameters has not been demonstrated adequately.

It is the purpose of this paper to present explicit correlations of eddy viscosities and Prandtl mixing lengths for the steady isothermal flow of constant-density fluids in smooth tubes. The effect of Reynolds number on the radial distribution of the two quantities has been calculated for both the transition and fully turbulent ranges of flow. The supporting data have been smoothed and made internally consistent in order to increase the ultimate utility of the calculated results.

BASIS OF CORRELATION

The steady, isothermal flow of a constant-density fluid through a long, straight tube of circular cross section will be considered, with u representing the mean temporal fluid velocity at the radial distance y from the tube wall and τg_0 the corresponding value of the mean local shearing stress. Following the procedure used by Murphree (*5*), the shearing stress for such a case of flow may be written in terms of the eddy viscosity ϵ as

$$\tau g_0 = (\mu + \epsilon)\frac{du}{dy} \tag{1}$$

The shearing stress might also be expressed in terms of the Prandtl mixing length (*9*). As shown by Schlichting (*12*), the relationship between the eddy viscosity and the mixing length l is

$$\epsilon = \rho l^2\left(\frac{du}{dy}\right) \tag{2}$$

If the fluid motion at the point under consideration is entirely viscous, the eddy viscosity and mixing length are zero.

Regardless of the prevailing type of flow, the local shearing stress varies linearly with the distance from the wall of the tube (*13*). Thus, if $\tau_0 g_0$ represents

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the shearing stress at the wall and r_0 the tube radius,

$$\tau g_0 = \tau_0 g_0\left(1 - \frac{y}{r_0}\right) \qquad (3)$$

Since the skin friction $\tau_0 g_0$ can be obtained readily from pressure-drop correlations, the cited equations permit calculation of the eddy viscosity and mixing length from experimental velocity distributions.

In the present work it has been found convenient to correlate the results in terms of two dimensionless groups, E and L, defined as follows:

$$E = \frac{\epsilon}{r_0 u_* \rho}\left(\frac{u_m}{V}\right)_p \qquad (4)$$

and

$$L = \frac{l}{r_0}\left(\frac{u_m}{V}\right)_p \qquad (5)$$

The symbol u_* denotes the friction velocity $\sqrt{\tau_0 g_0/\rho}$ The groups E and L have the same form as those used by Nikuradse (*6*) and others, except for the introduction of the velocity ratio $(u_m/V)_p$. The latter ratio was used by Rothfus and Monrad (*11*) in modifying the usual u^+, y^+ correlation of velocity distribution in order to remove the effect of Reynolds number in the fully turbulent range. They found that the relationship between the new parameters

$$U^+ = \frac{u}{u_*}\left(\frac{V}{u_m}\right)_p \qquad (6)$$

and

$$Y^+ = \frac{y u_* \rho}{\mu}\left(\frac{u_m}{V}\right)_p \qquad (7)$$

could be adequately represented by a single curve at Reynolds numbers greater than 3,000 in smooth tubes. It was also shown that the same correlation could be used for flow between parallel plates if $(V/u_m)_p$ were taken to be that in a tube having the same radius as the half clearance between the plates and operating at the same friction velocity u_*, with a fluid of the same kinematic viscosity μ/ρ as that between the plates.

In terms of the U^+ and Y^+ parameters, Equations (1) and (2) take the forms

$$E = \frac{(Y_m^+ - Y^+)}{(Y_m^+)^2(dU^+/dY^+)} - \frac{1}{Y_m^+}\left(\frac{u_m}{V}\right)_p^2 \qquad (8)$$

and

$$L = \left[\frac{E}{Y_m^+(dU^+/dY^+)}\right]^{1/2} \qquad (9)$$

The symbol Y_m^+ denotes the maximum (i.e., center-line) value of the modified friction distance parameter Y^+.

At very high Reynolds numbers the velocity distribution in the main stream can be represented approximately by an equation of the form

$$U^+ = A + B \ln Y^+ \qquad (10)$$

where A and B are constants. Under such conditions the effect of the viscous term in Equation (8) is negligible and Equations (8) and (9) reduce to the simple relationships

$$E = \frac{1}{B}\left(\frac{y}{r_0}\right)\left(1 - \frac{y}{r_0}\right) \qquad (11)$$

and

$$L = \frac{1}{B}\left(\frac{y}{r_0}\right)\sqrt{1 - \frac{y}{r_0}} \qquad (12)$$

To the extent that Equation (10) is correct, therefore, E and L should prove to be independent of Reynolds number; the maximum value of E should be obtained at $y/r_0 = 0.50$ and that of L at $y/r_0 = 0.67$. Since the logarithmic velocity distribution cannot be valid at the center of the stream or close to the wall of the tube, even at high Reynolds numbers, these conclusions can be taken only as rough approximations. They do, however, afford a starting point for any attempt to describe the behavior of the dimensionless groups E and L in the lower ranges of Reynolds number.

At high Reynolds numbers the thickness of the buffer layer and laminar film (if any such film exists) is a very small part of the distance from the center of the stream to the wall. For practical purposes the main-stream behavior can be taken to apply over the entire cross section of the fluid. At lower Reynolds numbers the buffer layer and laminar film are thicker. Therefore, the eddy viscosity and mixing length can be expected to be zero or very small for a measurable distance from the wall. In view of this, it is probable that the maximum point in the profile of E or L shifts toward the center of the stream as the Reynolds number is decreased through the lower turbulent and transition ranges of flow.

SOURCES OF VELOCITY DATA

It is apparent that accurate values of the velocity gradient must be available before the eddy viscosity and Prandtl mixing length can be computed satisfactorily from experimental velocity data. Suitable gradients can be obtained only when the velocity profile is established by numerous points of sufficiently high precision. This requirement makes it necessary to reject the results of several investigations which could profitably be included if velocity distribution alone were the subject of primary concern.

The velocity data of Nikuradse (*6*) and of Senecal and Rothfus (*14*) can be used for smooth tubes and those of Sage and coworkers (*1, 7, 8*) for flow between parallel plates. The Reynolds-number range included is 600 to 3,240,000 for tubes and 6,960 to 53,400 for parallel plates. The notable smooth-tube data of Deissler (*2*) can be used to establish velocity profiles but have not been presented in a form adaptable to the calculation of velocity gradients.

Nikuradse has presented profiles of the total effective viscosity as well as of a Prandtl type of mixing length calculated by means of the equation

$$l = u_* \sqrt{1 - y/r_0}/(du/dy) \qquad (13)$$

His tables contain several numerical errors, however, and his velocity gradients near the center of the tube appear to lack satisfactory precision. The graphs of mixing length against position in the stream indicate a consistent increase of l/r_0 with increasing y/r_0 at constant Reynolds number in the turbulent range.

Sage and coworkers (*8*) have calculated eddy-viscosity profiles from their excellent velocity data for flow between broad, parallel plates. The velocity distributions for tubes and parallel plates have been found to coincide under the conditions imposed by Rothfus and Monrad as noted in a previous paragraph. By virtue of Equations (8) and (9), therefore, the corresponding profiles of E and L must also be coincident on this basis. The data for parallel plates can thus be used in establishing the eddy-viscosity and mixing-length profiles for tubes as well.

The literature contains no explicit information about the characteristics of the eddy viscosity and mixing length in the range of laminar-turbulent transition. The velocity data of Senecal and Rothfus, however, provide the raw material necessary for the calculation of these profiles at tube Reynolds numbers up to 4,000.

COMPUTATIONS AND RESULTS

In both the transition and turbulent ranges of flow, the eddy viscosity and mixing length parameters were calculated by means of Equations (8) and (9) respectively. The technique used in establishing the proper value of the gradient dU^+/dY^+ was dependent on the precision of the experimental information available in the Reynolds number range under consideration.

The Transition Range

In the transition range, at Reynolds numbers from 1,200 to 4,000, the velocity data of Senecal and Rothfus were first plotted as curves of U^+ against Reynolds number at constant values of y/r_0 ($= Y^+/Y_m^+$). Cross plots of U^+ against y/r_0 at constant values of the Reynolds number were then constructed and the curves on both diagrams were adjusted to achieve smoothness, consistency, and the best agreement with the data. Figure 1 shows the final form of the velocity correlation.

The gradient (dU^+/dY^+) was obtained

from the derivative $dU^+/d(y/r_0)$, since by definition

$$\frac{dU^+}{dY^+} = \frac{1}{Y_m{}^+} \frac{dU^+}{d(y/r_0)} \qquad (14)$$

Values of U^+ were read from Figure 1 at even increments of 0.1 y/r_0 and difference tables were constructed from which values of $dU^+/d(y/r_0)$ were calculated numerically. Any one of three interpolation formulas, two by Newton and one by Stirling, was used in these calculations. The choice of a formula depended on its applicability in the particular range of y/r_0 being investigated. In regions of the stream where more than one formula could be used equally well, the average value of the gradient obtained by the various means was computed. The quantity E was then calculated by the use of Equations (8) and (14) for each point in the cross section at which the gradient had been determined. These values were smoothed with respect to both radial position and Reynolds number in a manner similar to that described for the velocity distribution data. The consistent, smoothed results for Reynolds numbers between 1,600 and 4,000 are shown in Figure 2. Corresponding values of L, calculated by means of Equation (9), are shown in Figure 4.

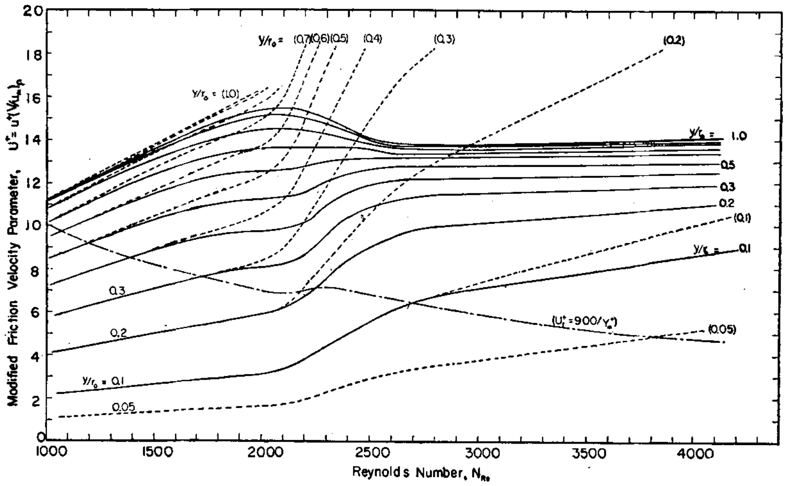

Fig. 1. Correlation of local velocities for transition flow in smooth tubes; dotted lines are calculated from viscous-flow equation (15).

The Turbulent Range

In the fully turbulent range, at Reynolds numbers between 4,000 and 3,240,000, values of the eddy viscosity parameter E were calculated largely from the velocity data of Nikuradse. The parallel-plate velocity data of Sage and his coworkers were taken to be equivalent to smooth-tube data on the basis proposed by Rothfus and Monrad. Upon appropriate transformation, values of E obtained from Sage's data were used to supplement those calculated from Nikuradse's results.

Nikuradse presented values of the velocity gradient du/dy but did not clearly indicate his method of obtaining the derivative. His tabulated results scattered sufficiently, however, to suggest that his method was not precise or that the velocity data were not smoothed before the derivatives were calculated. For more reliable values of the velocity gradient from Nikuradse's data, the following graphs were drawn at each Reynolds number investigated by him:

in the range

$$0.3 < y/r_0 < 1.0, \ \ln u \text{ against } (y/r_0)$$

in the range

$$0.02 < y/r_0 < 0.3, \quad u \text{ against } (y/r_0)$$

Straight lines representing the data as closely as possible were drawn on both diagrams. Curves of residual velocity showing the deviation of the data from the two straight lines as a function of radial position were than drawn and smoothed. The slopes of these curves were measured by means of a prismatic tangent meter. The value of the derivative $du/d(y/r_0)$ at a particular y/r_0 was then obtained by adding the slope of the residual curve to the slope of the appropriate straight line. The gradient du/dy followed directly. The values of du/dy thus obtained were checked by means of difference tables as previously described

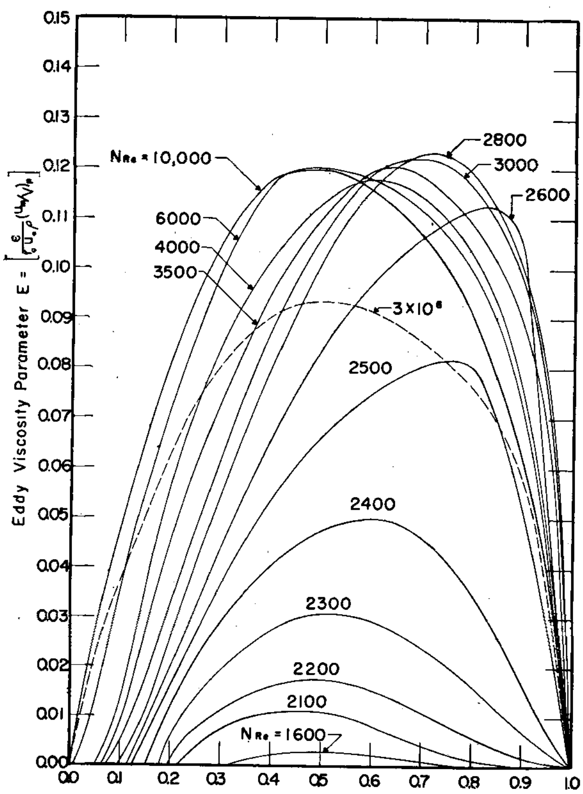

Fig. 2. Effect of Reynolds number on the radial distribution of the eddy-viscosity parameter E in the transition flow range.

for the transition region, and excellent agreement was noted.

In order to show the precision of the results and the nature of Reynolds-number dependence, values of E in both the transition and turbulent ranges of flow are plotted in Figure 3 as functions of Reynolds number at constant values of y/r_0. The solid lines represent the recommended correlation, smoothed with respect to both Reynolds number and radial position. In the placement of the recommended lines due consideration was given to the value of dU^+/dY^+ derivable from the generalized U^+, Y^+ correlation through the use of difference tables. Figure 3 can be used directly as a working graph at Reynolds numbers above 4,000. The nature of the correlation, however, makes it advisable to use Figure 2 in the transition range. For purposes of comparison, Figure 2 also contains curves showing the radial distribution of the eddy viscosity group at Reynolds numbers of 6×10^3, 1×10^4 and 3×10^6.

Figure 4 shows values of the mixing-length group L, calculated by means of Equation (9), for several Reynolds numbers in the transition range. The radial distribution at a Reynolds number of 3×10^6 is also included. In every case the lines correspond to the recommended values of E shown in Figures 2 and 3. To show a comparison with Nikuradse's data, points representing the average of his results for Reynolds numbers above 100,000 are included in Figure 4.

The values of the velocity ratio $(V/u_m)_p$ which were used in computing E and L are shown in Table 1. Obtained directly from experimental data, these represent the results of several reliable investigations (*3, 4, 6, 14, 15*).

DISCUSSION OF RESULTS

Figure 3 shows that the radial distribution of the eddy viscosity group E is not entirely independent of Reynolds number in the higher turbulent range. At Reynolds numbers greater than 100,000, however, the variation is small for y/r_0 values from 0.3 to 0.7. It is apparent that the effect of the $(V/u_m)_p$ ratio in the eddy viscosity group is to minimize the effect of Reynolds number in the central portion of the stream, where y/r_0 is greater than 0.7. At the same time values of E close to the wall are maintained reasonably independent of Reynolds number, the variation at $y/r_0 = 0.1$ being in the neighborhood of 10% over the range $10^5 < N_{Re} < 3 \times 10^6$.

Since viscous effects are negligible at high Reynolds numbers, the small variation of E with Reynolds number must be the result of a Reynolds-number effect on the U^+, Y^+ velocity correlation. This effect is imperceptible in the velocity distribution itself but is magnified when local gradients are taken. Even though the U^+, Y^+ correlation is not unique, no part of the main stream shows much effect of Reynolds number on E and L in the upper turbulent range of flow.

Nikuradse correlated eddy viscosities and mixing lengths in terms of the dimensionless groups $(\mu + \epsilon)/r_0 u_* \rho$ and l/r_0 respectively. The latter group was obtained from Equation (13) and therefore involves the mixing length defined by that equation. At high Reynolds numbers, where the contribution of viscous shear is negligible, Nikuradse's groups are almost independent of Reynolds number except near the center of the tube, where they increase with increased Reynolds number to a greater extent than do E and L. This simply reflects the small rotation of the lines in Figure 3 resulting from inclusion of the velocity ratio $(V/u_m)_p$ in the ordinate E.

The most striking departure from Nikuradse's conclusions is evident in the radial distribution of the mixing-length parameter near the center of the stream. His published results indicate that values of l/r_0 calculated from Equation (13) increase with distance from the wall all the way to the center of the tube. Careful reevaluation of his data, however, has led to the conclusion that the mixing-length parameter goes through a maximum point and tends strongly toward zero where the center of the stream is approached, as shown in Figure 4. The fact that the velocity gradients calculated in the present work are more consistent than Nikuradse's lends support to this conclusion. In the last analysis, however, it remains for more precise measurements of local velocities to furnish the basis for a firm judgement. It should be noted that the Prandtl and von Karman logarithmic velocity distributions predict zero mixing length at the center of the tube. Since both theories also indicate a finite velocity gradient at the center, they cannot be considered valid in the latter vicinity; however, they do predict qualitatively the relation between l/r_0 and y/r_0 which has been calculated from experimental data obtained in the central portions of smooth tubes.

At Reynolds numbers less than 10,000, there is a decided effect of Reynolds number on the magnitude and radial distribution of the parameters E and L. The nature of the variation is illustrated in Figures 2, 3, and 4. The values of E over the cross section of the tube increase rapidly as Reynolds number is increased above 1,600. At a Reynolds number of between 3,000 and 10,000, depending on the value of y/r_0, E reaches a maximum and then decreases with a further increase of N_{Re}. This behavior of E in the transition and lower turbulent-flow regions cannot with certainty be interpreted in terms of existing theories of turbulence.

The effect of laminar-film thickness in the transition zone is clearly indicated in Figures 2 and 4. The radial distributions of the eddy viscosity and mixing-length parameters are set out a greater and greater distance from the wall as the film thickens with decreased Reynolds number.

It is notable also that the points of maximum eddy viscosity and mixing length shift toward the center of the stream in the transition range. Particularly in the case of the mixing length, this is accompanied by an increase in the magnitude of the parameter L. Thus the central region of the tube in transition flow becomes a zone of large-scale mixing. Dye studies (*10*) indicate that the flow in this region may be sinuous at one instant and part of a large disturbance eddy at another. In either case, however, relatively long segments of the dye filament move as units without appreciable rapid small-scale diffusion; but it should be recognized that these low-frequency transient effects influence the film thickness, eddy viscosity, and mixing length in the transition range. The values indicated in the cited graphs are time averages and are therefore dependent on the frequency with which strong disturbance eddies are cast off and move downstream.

Figure 1 can be used as a working correlation of local velocities in the transition range of flow. This type of correlation aids interpolation, as it eliminates the crossing over of neighboring lines. The graph illustrates the complexity of correlation in the transition region and emphasizes the fact that in transition flow the functional relation between U^+ and Y^+ depends on the value of Reynolds number—unlike the relation in turbulent flow, where the relation between U^+ and Y^+ is virtually independent of Reynolds number.

The dashed lines in Figure 1 indicate the velocity distribution calculated by the equation

$$U^+ = \frac{Y_m^+}{2}\left(\frac{V}{u_m}\right)_p^2\left[\frac{Y^+}{Y_m^+}\left(1 - \frac{Y^+}{2Y_m^+}\right)\right]$$

$$= \frac{1}{2} N_{Re}\sqrt{\frac{f}{2}}\left(\frac{V}{u_m}\right)_p\left[\frac{y}{r_0}\left(1 - \frac{y}{2r_0}\right)\right] \quad (15)$$

This expression can be obtained from Equation (8) by assuming that $E = 0$ and integrating at a given Reynolds number from the tube wall, where U^+ and Y^+ are zero, to any arbitrary point in the fluid stream where $Y^+ = (y/r_0)(Y_m^+)$ and $U^+ = U^+$. In the calculations the actual values of f and $(V/u_m)_p$ for smooth tubes at the given Reynolds number were used. A U^+ calculated from Equation (15) should be the correct one in the regions adjacent to the wall, where the assumption that $E = 0$ is reasonably accurate. The profiles calculated by Equation (15) aid in interpreting the experimentally determined velocity distribution in the transition range. Since the actual profiles depart so gradually from the viscous distributions in the lower transition zone, it is difficult to establish the laminar-film thickness very precisely. But it does appear, however, that the modified friction velocity parameter U_f^+ taken at the edge of the laminar film is related to the center-line value Y_m^+ of the modified friction distance parameter through the expression

$$U_f^+ Y_m^+ = 900 \qquad (16)$$

within the limits of observation. Dye studies by Prengle and Rothfus (*10*) and pressure-drop data by Senecal and Rothfus (*14*) have yielded a value of about 1,200 for the constant in the last equation. In both cases the experimental technique might be expected to produce somewhat higher values than those obtained from velocity measurements. The agreement can therefore be considered satisfactory in view of the various uncertainties involved. Since the flow within the laminar layer is parabolic, the radial distance from the center of the stream to the edge of the laminar film r_f can be obtained from Equation (16) and is given by the relationship

$$1 - \left(\frac{r_f}{r_0}\right)^2 = \frac{14{,}400}{(N_{Re}\sqrt{f'})^2} \qquad (17)$$

Prengle and Rothfus report a value of 19,600 for the constant on the right-hand side of the equation.

It is apparent from Figure 1 that the divergence of the actual velocity profiles from the viscous-flow extrapolations become less pronounced at very small values of y/r_0. This suggests that a higher Reynolds numbers where the laminar film, if any, is extremely thin, velocity measurements become poor indicators of the film thickness. Instead, it can be predicted that such measurements should lead to the notion that a much thicker film exists than is actually the case, since the effects of weak eddies in the buffer layer next to the film will not be picked up by ordinary means. Consequently, it is not surprising that velocity data seem to indicate that the modified friction distance parameter Y_f^+ at the edge of the apparent film is approximately constant and independent of Reynolds number at Reynolds numbers above, say, 10,000.

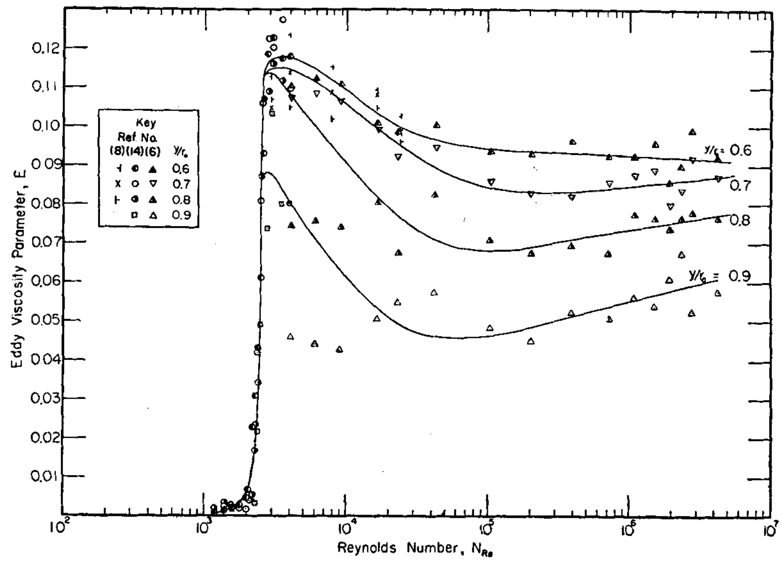

Fig. 3*a*.

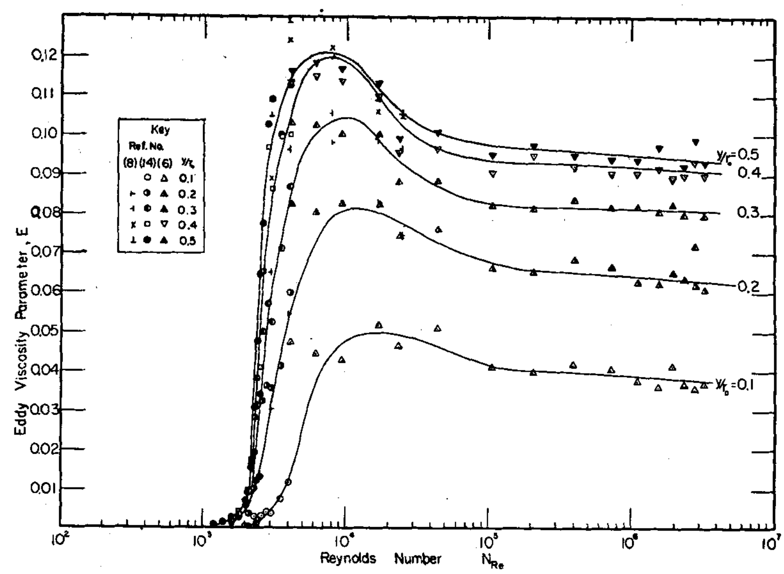

Fig. 3*b*.

Fig. 3. Effect of Reynolds number on the eddy-viscosity parameter E at fixed position in the stream.

TABLE 1

THE RATIO OF BULK AVERAGE TO MAXIMUM VELOCITIES IN SMOOTH TUBES

N_{Re}	$(V/u_m)_p$	N_{Re}	$(V/u_m)_p$
2,000	0.500	8,000	0.774
2,100	0.502	9,000	0.778
2,200	0.567	10,000	0.780
2,300	0.604	15,000	0.792
2,400	0.636	20,000	0.798
2,500	0.667	25,000	0.803
2,600	0.685	30,000	0.808
2,700	0.697	40,000	0.812
2,800	0.705	50,000	0.818
2,900	0.710	60,000	0.822
3,000	0.717	80,000	0.826
3,250	0.724	1×10^5	0.830
3,500	0.731	2×10^5	0.841
3,750	0.738	4×10^5	0.852
4,000	0.744	6×10^5	0.857
4,500	0.752	1×10^6	0.864
5,000	0.758	2×10^6	0.872
6,000	0.765	3×10^6	0.877
7,000	0.770		

The eddy viscosity and mixing-length parameters used in Figures 2, 3, and 4 were chosen because they yield relationships which are essentially independent of Reynolds number in the higher turbulent range and are relatively easy to interpolate. To aid in the placing of the recommended lines on these graphs, the generalized diagram of U^+ against Y^+ was used as a guide in the middle range of y/r_0 values. Since the assumption of a unique U^+, Y^+ relationship is only a rough approximation near the wall and at the center of the stream, the slope of the generalized U^+, Y^+ curve is not sufficiently accurate to be of much help in these zones.

In the lower turbulent range it is apparent that the recommended curves in Figure 3 do not coincide with Nikuradse's data near the center of the tube. It is believed that the data of Senecal and Rothfus are more reliable than Nikuradse's in this region. The recommended lines have, therefore, been drawn to progress smoothly to Senecal's values at the upper limit of the transition region.

The parallel-plate data of Sage and his coworkers were found to be in excellent agreement with tube data on the equivalent basis suggested by Rothfus and Monrad. This result was to be expected in view of the excellent agreement in velocity data obtained on U^+, Y^+ coordinates. The correspondence of eddy viscosities, however, constitutes a more stringent check on the postulated equivalence of tubes and parallel plates.

Isakoff and Drew (*3*) have measured velocity profiles for mercury flowing

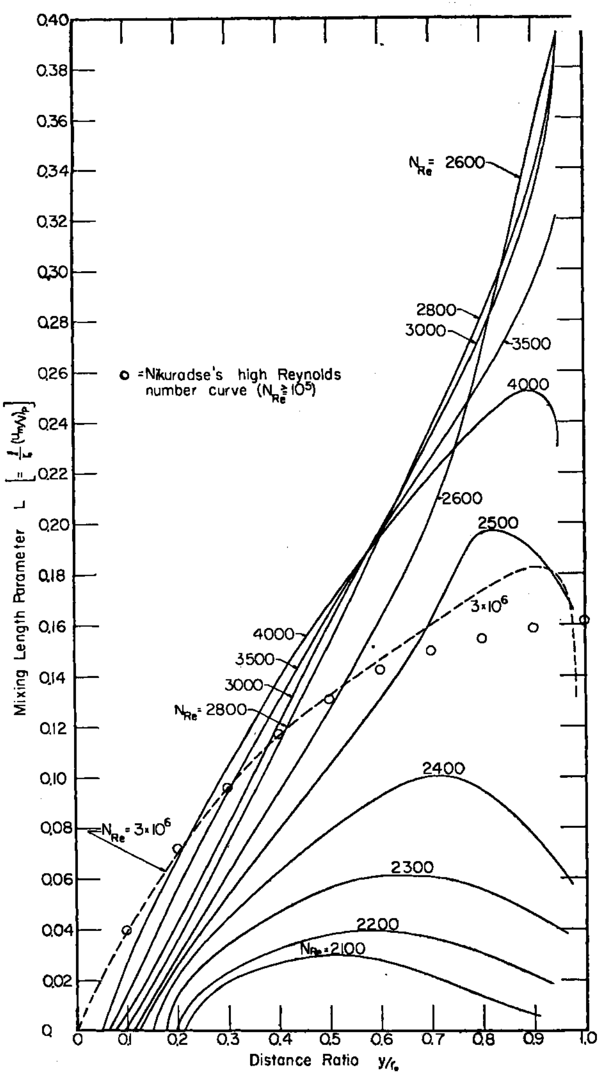

Fig. 4. Effect of Reynolds number on the radial distribution of the mixing-length parameter *L* in the transition flow range.

through smooth tubes at ten different Reynolds numbers in the range of 125,000 to 500,000. They present a graph which shows a calculated eddy momentum diffusivity ratio $(\epsilon/\rho)/(\epsilon/\rho)_{max} = E/E_{max}$ as a function of y/r_0. They were unable to detect a variation of this ratio with Reynolds number in the range of their investigation. Their calculated values of E/E_{max} agree with those presented in Figure 3 with an accuracy of $\pm 7\%$ except in the center of the tube, where the Isakoff and Drew calculations are about 15% greater than the results presented here. Since Isakoff and Drew state that their diffusivity data scatter by as much as $\pm 10\%$, it can be concluded that there is substantial agreement between their work and the results presented here. Furthermore they conclude that $(\epsilon/\rho)_{max}$ varies as the 0.83 power of the Reynolds number. Figure 3 predicts that for a given fluid $(\epsilon/\rho)_{max}$ will vary with about the 0.9 power of the Reynolds number over the range of Reynolds numbers investigated by Isakoff and Drew.

NOTATION

A, B = constants
E = eddy viscosity parameter $\epsilon/r_0 u_* \rho \; (u_m/V)_p$, dimensionless
f = fanning friction factor, dimensionless
g_0 = conversion factor, 32.2 $\frac{\text{(lb. mass)(ft.)}}{\text{(lb. force)(sec.}^2\text{)}}$
l = Prandtl mixing length, ft.
L = Prandtl mixing-length parameter $(l/r_0)(u_m/V)_p$, dimensionless
N_{Re} = Reynolds number for flow in a tube $2r_0V\rho/\mu$, dimensionless
r = distance to a point measured from the tube axis, ft.
r_f = distance from the tube axis to the edge of the laminar film, ft.
r_0 = tube radius, ft.
u_m = maximum flow velocity within the tube, ft./sec.
$(u_m/V)_p$ = ratio of the maximum to the average flow velocity in a smooth tube, dimensionless
u_* = friction velocity parameter $\sqrt{\tau_0 g_0/\rho}$, ft./sec.
u^+ = velocity parameter u/u_*, dimensionless
U^+ = modified velocity parameter $u^+ \, (V/u_m)_p$, dimensionless
V = average flow velocity within a tube, ft./sec.
x = distance measured along the tube axis, ft.
y = distance measured in a radial direction from the tube wall $(r_0 - r)$, ft.
y^+ = distance parameter $yu_*\rho/\mu$, dimensionless
Y^+ = modified dimensionless distance parameter $(u_m/V)_p \, y^+$, dimensionless
Y_m^+ = maximum value of Y^+ in a tube $(r_0u_*\rho/\mu) \; (u_m/V)_p$, dimensionless

Greek Letters

ϵ = eddy viscosity, lb. mass/(ft.)(sec.)
μ = coefficient of viscosity, lb. mass/(ft.)(sec.)
ρ = fluid density, lb. mass/cu. ft.
τ = shear stress at any point in the tube, lb. force/sq. ft.
τ_0 = shear stress at the tube wall, lb. force/sq. ft.

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